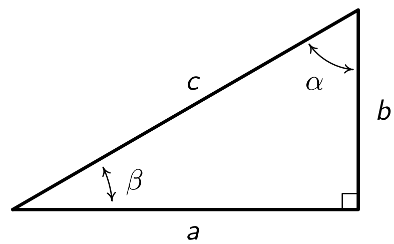
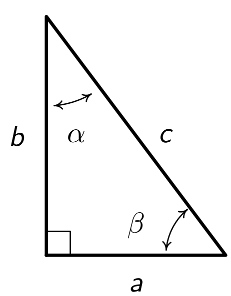
MATH 1700: SECTION B.2: RIGHT TRIANGLE TRIGONOMETRY

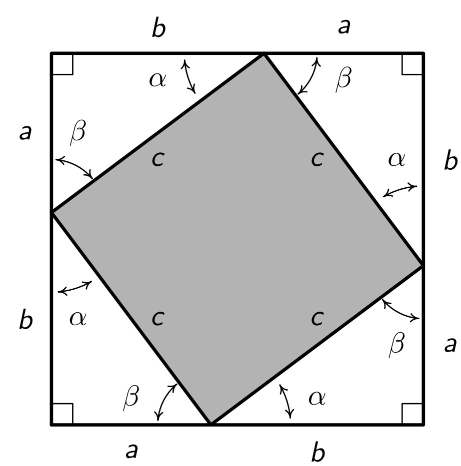
RIGHT TRIANGLES:

A **right triangle** is a triangle in which one angle measures 90°. Consider the right triangle below, where, as usual, the small square denotes the right angle, the labels 'a,' 'b,' and 'c' denote the lengths of the sides of the triangle, and α and β represent the (measure of) the non-right angles. As you may recall, the side opposite the right angle is called the **hypotenuse** of the right triangle. Also note that since the sum of the measures of all angles in a triangle must add to 180°, we have $\alpha + \beta + 90° = 180°$, or $\alpha + \beta = 90°$. Said differently, the non-right angles in a right triangle are *complements*.



The Pythagorean Theorem states that the square of the length of the hypotenuse of a right triangle is equal to the sums of the squares of the other two sides. So in our figure above, $a^2 + b^2 = c^2$. Below we sketch a proof.





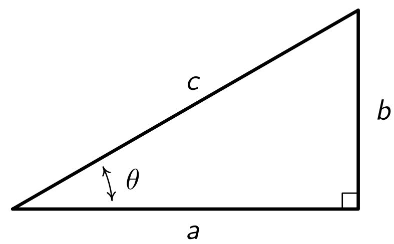
It should be noted that the converse of the Pythagorean Theorem is also true. That is if a, b, and c are the lengths of sides of a triangle and $a^2 + b^2 = c^2$, then c the triangle is a right triangle. (We'll prove this later in the course when we study the Law of Cosines.)

PYTHAGOREAN TRIPLES:

A list of integers (a, b, c) which satisfy the relationship $a^2 + b^2 = c^2$ is called a **Pythagorean Triple**. Some of the more common triples are: $(3, 4, 5)$, $(5, 12, 13)$, $(7, 24, 25)$, and $(8, 15, 17)$. We leave it to the reader to verify these integers satisfy the equation $a^2 + b^2 = c^2$ and suggest committing these triples to memory.

THE TRIGONOMETRIC RATIOS:

Given any acute angle θ, we can imagine θ being an interior angle of a right triangle as seen below.

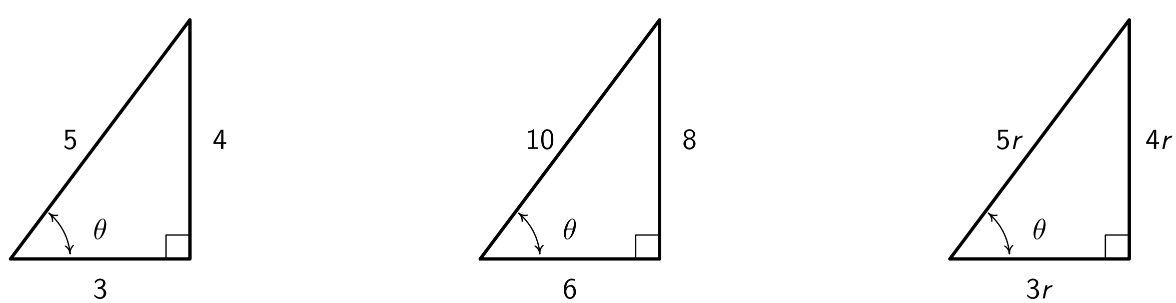


Focusing on the arrangement of the sides of the triangle with respect to the angle θ, we make the following definitions: the side with length a is called the side of the triangle which is **adjacent** to θ and the side with length b is called the side of the triangle **opposite** θ. As usual, the side labeled 'c' (the side opposite the right angle) is the hypotenuse. Using this diagram, we define three important **trigonometric ratios** of θ.

Suppose θ is an acute angle residing in a right triangle as depicted above.

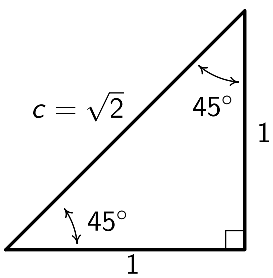
- The **sine** of θ, denoted $\sin(\theta)$ is defined by the ratio: $\sin(\theta) = \dfrac{b}{c}$, or $\dfrac{\text{'length of opposite'}}{\text{'length of hypotenuse'}}$.
- The **cosine** of θ, denoted $\cos(\theta)$ is defined by the ratio: $\cos(\theta) = \dfrac{a}{c}$, or $\dfrac{\text{'length of adjacent'}}{\text{'length of hypotenuse'}}$.
- The **tangent** of θ, denoted $\tan(\theta)$ is defined by the ratio: $\tan(\theta) = \dfrac{b}{a}$, or $\dfrac{\text{'length of opposite'}}{\text{'length of adjacent'}}$.

For example, consider the angle θ indicated in the triangle below on the left. We get $\sin(\theta) = \frac{4}{5}$, $\cos(\theta) = \frac{3}{5}$, and $\tan(\theta) = \frac{4}{3}$. Note that these ratios we've found for θ remain unchanged if the triangle containing θ changes. For example, if we scale all the sides of the triangle below on the left by a factor of 2, we produce the **similar triangle** below in the middle. Using this triangle to compute our ratios for θ, we find $\sin(\theta) = \frac{8}{10} = \frac{4}{5}$, $\cos(\theta) = \frac{6}{10} = \frac{3}{5}$, and $\tan(\theta) = \frac{8}{6} = \frac{4}{3}$. Note that the scaling factor, here 2, is common to all sides of the triangle, and, hence, cancels from the numerator and denominator when simplifying each of the ratios. In general, thanks to the Angle Angle Similarity Postulate, any two *right* triangles which contain our angle θ are similar which means there is a positive constant r so that the sides of the triangle are $3r$, $4r$, and $5r$ as seen above on the right. Hence, regardless of the right triangle in which we choose to imagine θ, $\sin(\theta) = \frac{4r}{5r} = \frac{4}{5}$, $\cos(\theta) = \frac{3r}{5r} = \frac{3}{5}$, and $\tan(\theta) = \frac{4r}{3r} = \frac{4}{3}$. Generalizing this same argument to any acute angle θ assures us that the ratios as described above are independent of the triangle we use.



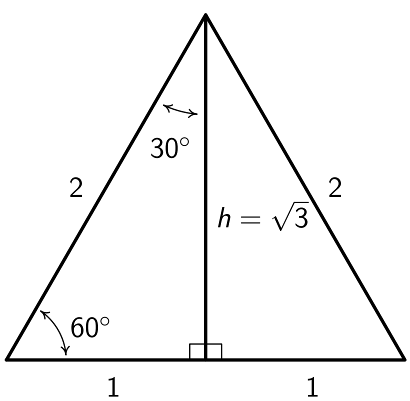
TRIGONOMETRIC RATIOS OF COMMON ANGLES:

Our next objective is to determine the values of $\sin(\theta)$, $\cos(\theta)$, and $\tan(\theta)$ for some of the more commonly used angles. We begin with 45°. In a right triangle, if one of the non-right angles measures 45°, then the other measures 45° as well. It follows that the two legs of the triangle must be congruent. Since we may choose any right triangle containing a 45° angle for our computations, we choose the length of one (hence both) of the legs to be 1. The Pythagorean Theorem gives the hypotenuse is: $c^2 = 1^2 + 1^2 = 2$, so $c = \sqrt{2}$. (We take only the positive square root here since c represents the length of the hypotenuse here, so, necessarily $c > 0$.) From this, we obtain the values below, and suggest committing them to memory.



- $\sin(45^\circ) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $\cos(45^\circ) = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $\tan(45^\circ) = \dfrac{1}{1} = 1$

Next, we investigate 60° and 30° angles. Consider the equilateral triangle below each of whose sides measures 2 units. Each of its interior angles is necessarily 60°, so if we drop an altitude, we produce two $30^\circ - 60^\circ - 90^\circ$ triangles each having a base measuring 1 unit and a hypotenuse of 2 units. Using the Pythagorean Theorem, we can find the height, h of these triangles: $1^2 + h^2 = 2^2$ so $h^2 = 3$ or $h = \sqrt{3}$. Using these, we can find the values of the trigonometric ratios for both 60° and 30°. Again, we recommend committing these values to memory.



- $\sin(60^\circ) = \dfrac{\sqrt{3}}{2}$
- $\cos(60^\circ) = \dfrac{1}{2}$
- $\tan(60^\circ) = \dfrac{\sqrt{3}}{1} = \sqrt{3}$
- $\sin(30^\circ) = \dfrac{1}{2}$
- $\cos(30^\circ) = \dfrac{\sqrt{3}}{2}$
- $\tan(30^\circ) = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

Since 30° and 60° are complements, the side *adjacent* to the 60° angle is the side *opposite* the 30° and the side *opposite* the 60° angle is the side *adjacent* to the 30° . We'll see this relationship generalized later.

THREE MORE RATIOS:

- The **cosecant** of θ, denoted $\csc(\theta)$ is defined by the ratio: $\csc(\theta) = \dfrac{c}{b}$, or $\dfrac{\text{'length of hypotenuse'}}{\text{'length of opposite'}}$.
- The **secant** of θ, denoted $\sec(\theta)$ is defined by the ratio: $\sec(\theta) = \dfrac{c}{a}$, or $\dfrac{\text{'length of hypotenuse'}}{\text{'length of adjacent'}}$.
- The **cotangent** of θ, denoted $\cot(\theta)$ is defined by the ratio: $\cot(\theta) = \dfrac{a}{b}$, or $\dfrac{\text{'length of adjacent'}}{\text{'length of opposite'}}$.

There are some natural relationships between these three 'new' ratios and the original three given here which we'll explore in great detail later. For instance, both $\sin(\theta)$ and $\csc(\theta)$ are ratios involving the length of the 'opposite' side and the the length of the hypotenuse. Which ratio is used comes down often to convenience.

EXAMPLE 1: Suppose $\sec(\theta) = 5$. Find the **exact** values of the remaining five trigonometric ratios of θ.

EXAMPLE 2: From an observation tower 200 feet off the ground, a ranger spies a Sasquatch. If the angle of declination from the tower to the 'squatch is $42°$, how far is the 'squatch from the base of the tower?

EXAMPLE 3: To measure the height of Godzilla, two sightings are taken one 100 feet behind the other. If the first angle of elevation is 37° and the second is 31°, what is the height of Godzilla?

FINDING ANGLES:

We can use 'inverse' or 'arc' functions to help us find angles from trigonometric ratios. The notation $\sin^{-1}(x)$ or $\arcsin(x)$ gives an angle θ where $\sin(\theta) = x$. We'll have much more to say about these functions later, but we introduce them here now since they are of practical importance.

EXAMPLE 4: Find the angles in a $3-4-5$ right triangle rounded to the nearest degree.

EXAMPLE 5: The roof on the house below has a '6/12 pitch'. This means that when viewed from the side, the roof line has a rise of 6 feet over a run of 12 feet. Find the angle of inclination from the bottom of the roof to the top of the roof. Round your answer to the nearest hundredth of a degree.

HOMEWORK: Appendix B.2: Page 1409: 1 - 29 odd, 32, 33